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# Differential Equations for Longitudinal Motion in a Synchrotron

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**Abstract**

Starting from finite difference equations of motion for the longitudinal phase space coordinates $(\varphi, W) = (\phi - \phi_{\mathrm{synch}}, [E - E_{\mathrm{synch}}]/\omega_{\mathrm{synch}})$ one can derive differential equations of motion which represent the same physical system except that the force from the rf is distributed uniformly over the full azimuth of the synchrotron. Because the difference equations are essentially free of kinematic or dynamic approximations, the resulting differential equations contain a more complete description of the dynamics than those usually employed. They may be used, for example, at or near the transition energy or for particles far from the synchronous energy. A brief discussion comparing these equations to the conventional ones is offered in conclusion.

## Introduction

The complete description of the single-particle motion of beam particles in a synchrotron requires specifying a trajectory in a six-dimensional space with time or, perhaps position around the ring, as a parameter. However, because transverse focusing for beam containment is typically $\sim 10^8$ times stronger than the phase focusing provided by the accelerating rf field, the characteristic frequencies of transverse and longitudinal oscillations differ by a factor $\sim 10^4$. Therefore, a small change in longitudinal coordinates occurs over many periods of the transverse oscillation, and only average properties of transverse potential have significant effect on what happens in the longitudinal coordinates. So, while it is certainly possible to write equations of motion starting from the general Hamiltonian for a charged particle in an electromagnetic field,[1] it is not generally done. The transverse and longitudinal motion are considered separately from the start; the choice of coordinates and momenta is specialized for convenience in the regime of practical accelerators. There are available many derivations in this vein of differential equations appropriate to bunched beams.[2][3][4] They are written in various forms; a common one, and one common to the cited references, is popular because it is Hamilton's equations for a simple Hamiltonian, *viz.*,

$$\dot{\varphi} = \frac{\omega_s^2 \eta}{\beta_s^2 E_s} W \tag{1}$$

$$\dot{W} = \frac{e\hat{V}}{2\pi h}[\sin(\varphi + \phi_s) - \sin(\phi_s)] \ , \tag{2}$$

where $\varphi$ is phase relative to the synchronous particle and $W$ is the energy difference divided by the synchronous angular frequency $\omega_s = h\Omega_s$. Several approximations are needed to derive these equations, but they are sometimes considered "exact throughout the acceleration cycle."[4] The standard treatments assume that the interesting regime is oscillatory motion of beam particles with respect to the trajectory of the synchronous particle. However, in accelerators and especially in some storage rings there are other important types of longitudinal motion. Examples include adiabatic capture, phase displacement, moving bucket capture, off-energy injection, slip stacking,[5] and many more.

In a previous note[6] the equations of longitudinal motion were derived as difference equations. For the purposes of numerical simulation of beam behavior, difference equations are a desirable mathematical framework because they translate immediately into arithmetic operations. It is also true, however, that difference equations are to be preferred conceptually because the rf force is applied impulsively at localized rf gaps so that the particle receives a finite energy increment followed by a finite rf-free drift. The difference step is taken as the time or phase between successive gaps. In the cited note the difference equations are written without recourse to kinematic or dynamic approximation. Thus, they are suitable as a step in the derivation of differential equations more general than eqs. 1 and 2.

The equations from ref. [6] are

$$\varphi_{i,n} = \frac{\omega_{s,n}}{\omega_{s,n-1}}\varphi_{i,n-1} + 2\pi h(S_{i,n} - 1) \tag{3}$$

$$W_{i,n} = \frac{\omega_{s,n-1}}{\omega_{s,n}} W_{i,n-1} + \frac{e}{\omega_{s,n}}[V(\varphi_{i,n} + \phi_{s,n}) - V(\phi_{s,n})] \ . \tag{4}$$

These equations may be understood without the explicit definition of every quantity by the following statements:

$i$ refers to a particle of interest.
$s$ refers to the synchronous particle.

$n$ refers to the $n$-th circulation around the ring ended by the $n$-th energy kick.
The rf frequency is $h$ times the circulation frequency of the synchronous particle; $h$ is an integer.
$\omega = h\Omega = hv/R$.
$\phi$ is the phase of the rf at the time the particle passes the gap.
$\varphi_i = \phi_i - \phi_s$.
$W_i = (E_i - E_s)/\omega_s$, where the $E$'s are total energy.
$S_i = \omega_s/\omega_i$ is the phase slip per turn.

They can be read as a map $\mathbf{M}$ of the point $(\varphi_{i,n-1}, W_{i,n-1})$ to another point$(\varphi_{i,n}, W_{i,n})$ in the $\varphi - W$ plane. The map is area preserving as one can determine from a direct calculation of the Jacobian derivative

$$J(\mathbf{M}) = \frac{\partial(\varphi_{i,n}, W_{i,n})}{\partial(\varphi_{i,n-1}, W_{i,n-1})} = \begin{vmatrix} \frac{\omega_{s,n}}{\omega_{s,n-1}} & \frac{e}{\omega_{s,n-1}} eV' \\ 2\pi h \frac{\partial S_{i,n}}{\partial W_{i,n-1}} & \frac{\omega_{s,n-1}}{\omega_{s,n}} + \frac{e}{\omega_{s,n}} V' 2\pi h \frac{\partial S_{i,n}}{\partial W_{i,n-1}} \end{vmatrix} \equiv 1 \ . \quad (5)$$

In ref. [6] the role of the small amount of betatron acceleration occurring in a synchrotron is discussed. Its influence is shown to be very small, and an argument given by Dôme[3] is cited to justify ignoring it in practice. However, contrary to that discussion and in distinction to Dôme's treatment one can evaluate the betatron acceleration consistently in the finite difference framework. Before deriving the differential equations it is demonstrated in the next section that the correct difference equations have no explicit betatron acceleration term.

# Betatron Acceleration

The per-turn betatron acceleration contribution to the difference $\varepsilon_{i,n} = E_{i,n} - E_{s,n}$ between the energy of particle $i$ and the synchronous particle is

$$\delta^{(\beta)}\varepsilon_{i,n} = -e \oint \dot{B}_z(r_{i,n} - r_{s,n}) \, dr \, d\theta \ , \quad (6)$$

where the polar cylindrical coordinates have their origin at the center of the ring and the integration extends over the area between the orbits. Throughout the following the operator $\delta$ denotes the difference between the same quantity on successive turns and the operator $\Delta$ denotes the difference between corresponding quantities for particle $i$ and particle $s$ on the same turn. Turns have differing periods for different particles, so a $\Delta$ difference is *not* an equal-time difference. The evaluation of $\delta^{(\beta)}$ is shown in detail to make the implicit assumptions evident.

Writing the radial separation of the orbits in terms of the dispersion $D$ and the momentum difference,

$$\delta^{(\beta)}\varepsilon_{i,n} = -e\oint \frac{\partial B_{z,s}}{\partial t}\,\Delta r\, r_s\, d\theta = -e\frac{\Delta p}{p_s}\oint \frac{\partial B_{z,s}}{\partial t}\, D(\theta)\, r_s\, d\theta \ . \tag{7}$$

Momentum and magnetic field are related by

$$|p_s| = \frac{e}{c}(B\rho)_s \ . \tag{8}$$

Therefore,

$$-\langle B_{z,s}\rangle = \frac{cp_s}{e}\langle\frac{1}{\rho}\rangle = \frac{cp_s}{e}\frac{1}{2\pi R_s}\oint\frac{ds}{\rho} = \frac{cp_s}{eR_s} \ . \tag{9}$$

Use this result in eq. 7 to get the $\dot{B}$ factor out of the integral:

$$\begin{aligned}\delta^{(\beta)}\varepsilon_{i,n} &= -e\frac{\partial\langle B_{z,s}\rangle}{\partial t}R_s\frac{\Delta p}{p_s}\oint D(\theta)\, d\theta = -e\frac{\partial\langle B_{z,s}\rangle}{\partial t}R_s\frac{\Delta p}{p_s}(2\pi\alpha_p R_s)\\ &= -e\frac{\partial\langle B_{z,s}\rangle}{\partial t}2\pi R_s\Delta R \end{aligned}\tag{10}$$

as one would write immediately for a model of the ring with an azimuthally uniform magnetic field. From the magnetic field *vs.* momentum relation eq. 9,

$$\dot{p}_s = -\frac{e}{c}\left[\frac{\partial\langle B_{z,s}\rangle}{\partial t}R_s + R_s\frac{\partial\langle B_{z,s}\rangle}{\partial R_s}\frac{dR_s}{dt} + \langle B_{z,s}\rangle\frac{dR_s}{dt}\right] \ , \tag{11}$$

and

$$\frac{\dot{p}_s}{p_s} = \frac{1}{\langle B_{z,s}\rangle}\frac{\partial\langle B_{z,s}\rangle}{\partial t} + \left[\frac{R_s}{\langle B_{z,s}\rangle}\frac{\partial\langle B_{z,s}\rangle}{\partial R_s} + 1\right]\frac{\dot{R}_s}{R_s} \ . \tag{12}$$

The term in square brackets is related to the momentum compaction $\alpha_p$ by[2]

$$\left[\frac{R_s}{\langle B_{z,s}\rangle}\frac{\partial\langle B_{z,s}\rangle}{\partial R_s} + 1\right] = \frac{1}{\alpha_p} \tag{13}$$

so that

$$\frac{\dot{p}_s}{p_s} = \frac{1}{\langle B_{z,s}\rangle}\frac{\partial\langle B_{z,s}\rangle}{\partial t} + \frac{1}{\alpha_p}\frac{\dot{R}_s}{R_s} \tag{14}$$

and

$$\frac{\partial\langle B_{z,s}\rangle}{\partial t} = \langle B_{z,s}\rangle\left[\frac{\dot{p}_s}{p_s} - \frac{1}{\alpha_p}\frac{\dot{R}_s}{R_s}\right] \ . \tag{15}$$

Replacing the $\dot{B}$ term in eq. 10 with this last expression one finds

$$\begin{aligned}
\delta^{(\beta)}\varepsilon_{i,n} &= -e\langle B_{z,s}\rangle \left[\frac{\dot{p}_s}{p_s} - \frac{1}{\alpha_p}\frac{\dot{R}_s}{R_s}\right] 2\pi R_s \Delta R = \left[\alpha_p \frac{\dot{p}_s}{p_s} - \frac{\dot{R}_s}{R_s}\right] \frac{2\pi p_s}{\alpha_p} \Delta R \\
&= \left[\alpha_p \frac{\dot{p}_s}{p_s} - \frac{\dot{R}_s}{R_s}\right] 2\pi R_s \Delta p \ . \qquad (16)
\end{aligned}$$

The $\Delta p$ throughout the course of the $n$-th turn is

$$\Delta p = v_s \Delta E = v_s \varepsilon_{i,n-1} \qquad (17)$$

so that

$$\delta^{(\beta)}\varepsilon_{i,n} = \frac{2\pi R_{s,n}\varepsilon_{i,n-1}}{v_{s,n}} \left[\alpha_p \frac{\dot{p}_s}{p_s} - \frac{\dot{R}_s}{R_s}\right] = \tau_{s,n} \left[\alpha_p \frac{\dot{p}_s}{p_s} - \frac{\dot{R}_s}{R_s}\right] \varepsilon_{i,n-1} \ . \qquad (18)$$

For any credible rate of change the time derivatives of synchronous momentum and radius are represented to very high accuracy by the difference quotients

$$\begin{aligned}
\dot{p}_s &= \delta p_s / \tau_s \\
\dot{R}_s &= \delta R_s / \tau_s \ . \qquad (19)
\end{aligned}$$

Therefore,

$$\delta^{(\beta)}\varepsilon_{i,n} = \left[\alpha_p \frac{\delta p}{p_{s,n}} - \frac{\delta R}{R_{s,n}}\right] \varepsilon_{i,n-1} = \left[\frac{\delta R}{R_{s,n}} - \frac{\delta R}{R_{s,n}}\right] \varepsilon_{i,n-1} = 0 \ . \qquad (20)$$

This result does not mean that there is no betatron acceleration. Rather it means that, if $eV(\phi_s)$ is set to $\delta E_s$ with betatron contribution included, there is no explicit contribution of the betatron acceleration to the difference between the synchronous and neighboring particles. This might be thought of as renormalizing $\phi_s$ to replace the betatron acceleration by an equivalent amount of rf acceleration. Note, however, that this result is only as good as the approximation of differences between quantities for particle $i$ and particle $s$ by first order differentials. Therefore, it is less general than the difference equations themselves. One should bear in mind that for some atypical situation it could be appropriate to carry along the explicit betatron acceleration contribution even though it would undoubtedly be small in a synchrotron.

# The Differential Equations

With the exception of the caveat from the previous section, eqs. 3 and 4 are equations of longitudinal motion essentially without approximation. Rewrite them with $\delta$'s as

$$\delta\varphi_{i,n} = \left(\frac{\omega_{s,n}}{\omega_{s,n-1}} - 1\right)\varphi_{i,n-1} + 2\pi h(S_{i,n} - 1) \tag{21}$$

$$\delta W_{i,n} = \left(\frac{\omega_{s,n-1}}{\omega_{s,n}} - 1\right) + \frac{e}{\omega_{s,n}}[V(\varphi_{i,n} + \phi_{s,n}) - V(\phi_{s,n})] \ . \tag{22}$$

The plan is to make time derivatives from the lefthand sides of these equations by dividing by the synchronous circulation period $\tau_{s,n}$. However, one could certainly object that the finite approximation does not represent the differential beyond first order. Therefore, the equations will be generalized slightly to the case of $N$ equally spaced cavities per turn, each having voltage $V/N$, all phased to $\phi_{s,n}$, and separated by drifts of equal average dispersion. The iteration of the map will now take the particle from one cavity to the next so that $n$ is no longer the turn index. The differential equations will be obtained by distributing the rf smoothly around the ring, *i. e.*, by taking in classical fashion $\lim N \to \infty$. This may or may not be a physically important approximation, but one should note that it is an approximation; the real equations of motion are the difference equations. For the $N$ cavities,

$$\delta\varphi_{i,n} = \left(\frac{\omega_{s,n}}{\omega_{s,n-1}} - 1\right)\varphi_{i,n-1} + \frac{2\pi h}{N}(S_{i,n} - 1) \tag{23}$$

$$\delta W_{i,n} = \left(\frac{\omega_{s,n-1}}{\omega_{s,n}} - 1\right)W_{i,n-1} + \frac{e}{N\omega_{s,n}}[V(\varphi_{i,n} + \phi_{s,n}) - V(\phi_{s,n})] \ . \tag{24}$$

Dividing both equations by the interval $\tau_{s,n}/N$ one gets

$$\frac{\delta\varphi_{i,n}}{\tau_{s,n}/N} = \frac{N}{\tau_{s,n}}\left(\frac{\omega_{s,n}}{\omega_{s,n-1}} - 1\right)\varphi_{i,n-1} + \omega_{s,n}(S_{i,n} - 1) \tag{25}$$

$$\frac{\delta W_{i,n}}{\tau_{s,n}/N} = \frac{N}{\tau_{s,n}}\left(\frac{\omega_{s,n-1}}{\omega_{s,n}} - 1\right)W_{i,n-1} + \frac{e}{2\pi h}[V(\varphi_{i,n} + \phi_{s,n}) - V(\phi_{s,n})] \ . \tag{26}$$

Now the time interval is arbitrary because $N$ is arbitrary. Therefore, exactly,

$$\begin{aligned}
\dot{\varphi}_i &= \underset{N \to \infty}{\text{Limit}}\left\{\frac{N}{\tau_{s,n}}\frac{\delta\omega_{s,n}}{\omega_{s,n}(1 - \delta\omega_{s,n}/\omega_{s,n})}\varphi_{i,n-1} + \omega_{s,n}(S_{i,n} - 1)\right\} \\
&= \frac{\dot{\omega}_s}{\omega_s}\varphi_i + \omega_s(S_i - 1) & (27) \\
\dot{W}_i &= \underset{N \to \infty}{\text{Limit}}\left\{-\frac{N}{\tau_{s,n}}\frac{\delta\omega_{s,n}}{\omega_{s,n}}W_{i,n-1} + \frac{e}{2\pi h}[V(\varphi_{i,n} + \phi_{s,n}) - V(\phi_{s,n})]\right\} \\
&= -\frac{\dot{\omega}_s}{\omega_s}W_i + \frac{e}{2\pi h}[V(\varphi_i + \phi_s) - V(\phi_s)] \ , & (28)
\end{aligned}$$

where the $n$ subscript gets dropped to indicate that the variables are continuous in the limit.

The differential equations have been derived from an area-preserving map **M**. If all has gone well in the derivation, the differential map d**M** for the transformation $(\varphi, W) \rightarrow (\varphi + d\varphi, W + dW)$ in time $dt$

$$\varphi_1 - \varphi_\circ = d\varphi = \left[\frac{\dot{\omega}_s}{\omega_s}\varphi_\circ + \omega_s(S-1)\right] dt \tag{29}$$

$$W_1 - W_\circ = dW = \left[-\frac{\dot{\omega}_s}{\omega_s}W_\circ + \frac{e}{2\pi h}[V(\varphi_\circ + \phi_s) - V(\phi_s)]\right] dt \tag{30}$$

should also be area preserving. The Jacobian is

$$J(\mathrm{dM}) = \frac{\partial(\varphi_1, W_1)}{\partial(\varphi_\circ, W_\circ)} = \begin{vmatrix} 1 + \frac{\dot{\omega}_s}{\omega_s}dt & \frac{eV'}{2\pi h}dt \\ \omega_s\frac{\partial S}{\partial W_\circ}dt & 1 - \frac{\dot{\omega}_s}{\omega_s}dt \end{vmatrix} = 1 - \mathcal{O}((dt)^2) \ . \tag{31}$$

The discrepancy in $J$ is of higher order than the precision of the differential map, but neglecting the order in which $\varphi$ and $W$ are incremented results in no cancelation of the off-diagonal elements in the determinant.

If the $\dot{\omega}_s/\omega_s$ terms were absent in eqs. 29 and 30, the Jacobian would be still be one to $\mathcal{O}(dt)$. These terms are required to preserve the phase space area in the finite mapping; they cancel to $\mathcal{O}(dt)$ in the Jacobian for the differential mapping but fail to cancel to $\mathcal{O}((dt)^2)$ because the terms depending on the order of the mapping steps are missing from the off-diagonal elements. The implication is that there is a Hamiltonian which contains a term to generate the $\dot{\omega}_s/\omega_s$ terms in the differential equations and an approximate Hamiltonian which does not. We can write the Hamiltonian for eqs. 27 and 28 formally as

$$H(\varphi, W) = \omega_s \int_0^W [S(w) - 1]\, dw - \frac{e\hat{V}}{2\pi h}[f(\varphi + \phi_s) - f(\phi_s) - \varphi f'(\phi_s)] + \frac{\dot{\omega}_s}{\omega_s}\varphi W \ , \tag{32}$$

where the potential is written as $\hat{V}g(\phi)$ and $f'(\phi) = g$. Although the equations derived from eq. 32 omitting the bilinear term are canonical equations of motion, the complete Hamiltonian is more faithful to the physical system. There may be circumstances in which this term is not negligible. Making $H$ an explicit function of $W$ depends on expanding

$$S - 1 = \frac{1 + \Delta R/R}{1 + \Delta\beta/\beta} - 1 \ . \tag{33}$$

One can identify two kinds of non-linearity, *viz.*, kinematic from expanding the denominator and lattice non-linearity from the dependance of $\Delta R/R$ on $\Delta p/p$. Either or both can be negligible depending on the parameter domain of interest. Of course for calculation one can evaluate $S = \Omega_s/\Omega$ exactly. For high $\beta$ it is reasonable to take only the first order $\Delta p/p$ term in the denominator so that

$$S - 1 = \frac{1 + \alpha_0 \frac{\Delta p}{p} + \alpha_1 \left(\frac{\Delta p}{p}\right)^2 + \cdots}{1 + \gamma^{-2} \frac{\Delta p}{p}} - 1 \ . \tag{34}$$

If one can ignore lattice non-linearity also, one finds with $\alpha_0 = \gamma_T^{-2}$ that

$$S - 1 \approx (\gamma_T^{-2} - \gamma^{-2}) \frac{\Delta p}{p_s} = \frac{\eta \omega_s}{\beta_s^2 E_s} W \ . \tag{35}$$

One arrives at eqs. 1 and 2 by neglecting the small $\dot{\omega}_s/\omega_s$ terms also.

# Conclusion

Differential equations of motion for the longitudinal coordinates in a synchrotron have been developed here which are more accurate (complete) than those found generally in standard sources. A treatment similar to this one has been given by Takayama[7] as part of a discussion of the dynamics near transition; various approximations he makes would be inappropriate in other circumstances. The derivation has been carried out with typical assumptions. Thus, a limitation to a system with an explicitly time-independent Hamiltonian has meant that certain contributions frequently of practical importance were not even considered, contributions for example from $\dot{V}$ and $\dot{\phi}_s$. These terms do not fit into the usual Hamiltonian formulation but can be derived easily from difference equations in the fashion employed in this note. The dominant theme here has been to isolate where the usual approximations are made and to indicate how they can be removed if necessary.

As one might expect from the examination of a topic frequently revisited, there is little in this treatment which is truly novel. However, the emphasis is different, and the use of difference equations free of differential approximations makes possible a systematic identification of error terms. The reader's attention has been drawn to $\dot{\omega}_s/\omega_s$ terms in the differential equations which are commonly neglected. The betatron acceleration contribution has been reexamined with a finite difference approach to establish the validity of the underlying difference equations. It may be that the Hamiltonian eq. 32 is novel at least in its explicit form.

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